Linear Regression Models with dependent errors

Emmanuel Caron [1]

http://ecaron.perso.math.cnrs.fr

Directeurs de thèse: Jérôme Dedecker (MAP5), Bertrand Michel (LMJL)

29 Mars 2019

[1] Ecole Centrale Nantes, Laboratoire de Mathématiques Jean Leray UMR 6629, 1 Rue de la Noë, 44300 Nantes. **Email: emmanuel.caron@ec-nantes.fr**

Sommaire

Sommaire

Time Series: CO2

$$Y_t = \underbrace{trend + seasonality}_{deterministic} + \underbrace{errors}_{random}$$

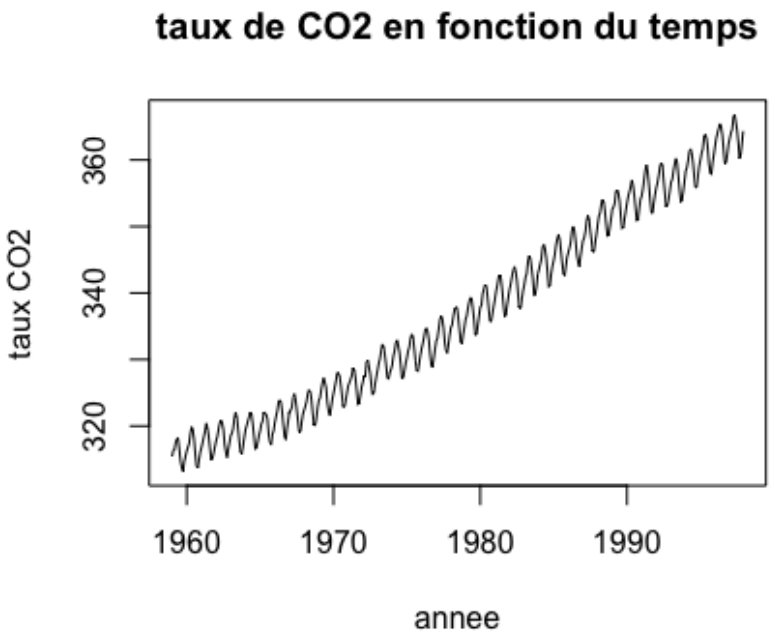


Figure : Time series: CO2

Linear Regression Model

$$Y = X\beta + \epsilon$$

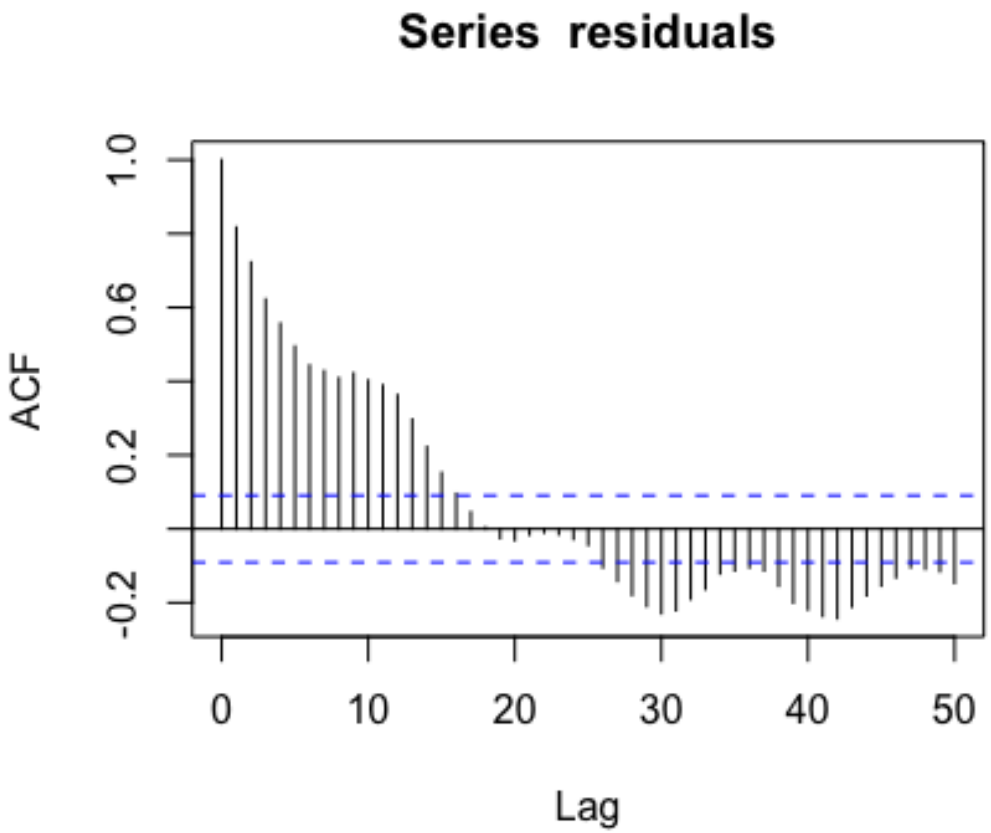
$$X = \begin{pmatrix} 1 & 1^2 & 1^3 & \cos(\frac{2\pi}{3}) & \sin(\frac{2\pi}{3}) & \dots & \cos(\frac{2\pi}{12}) & \sin(\frac{2\pi}{12}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ t & t^2 & t^3 & \cos(\frac{2\pi t}{3}) & \sin(\frac{2\pi t}{3}) & \dots & \cos(\frac{2\pi t}{12}) & \sin(\frac{2\pi t}{12}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ n & n^2 & n^3 & \cos(\frac{2\pi n}{3}) & \sin(\frac{2\pi n}{3}) & \dots & \cos(\frac{2\pi n}{12}) & \sin(\frac{2\pi n}{12}) \end{pmatrix}$$

$\hat{\beta} = (X^t X)^{-1} X^t Y$: Least Squares Estimators

ϵ: error process

$\hat{\epsilon} = Y - \hat{Y}$: residuals

ACF of the residuals



Figure

Goals and Plan

Main Goal : Study the linear regression model with dependent errors. Correct the results on this model in a very general framework.

Plan :

1. Framework
2. Hannan's Theorem (1973): convergence of the LSE in the stationary case under very mild conditions
3. Estimation of the covariance matrix
4. Application with Fisher's tests

Sommaire

Linear Model and Least Squares Estimator

$$Y = X\beta + \epsilon,$$

- X is a design, random or not, size $[n \times p]$
- Y is a n random vector
- β is a p vector of unknown parameters
- ϵ are the errors, $\epsilon \in \mathbb{R}^n$. The error process is independent of the design X.

$$\hat{\beta} = argmin_{\beta \in \mathbb{R}^p} \|Y - X\beta\|_2^2 = (X^t X)^{-1} X^t Y.$$

- $\hat{Y} = X\hat{\beta}$: Orthogonal Projection of Y on $\mathcal{M}_X = Vect\{X_{.,1}, ..., X_{.,p}\}$
- Residual vector: $\hat{\epsilon} = Y - \hat{Y} = Y - X\hat{\beta} \in \mathcal{M}_X^{\perp}$
- $\hat{\sigma}^2 = \frac{\|\hat{\epsilon}\|_2^2}{n-p}$.

Stationarity

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. $(\epsilon_i)_{i\in\mathbb{Z}}$ is an error process defined on $(\Omega, \mathcal{F}, \mathbb{P})$, supposed strictly stationary, with zero mean, and $\epsilon_0 \in \mathbb{L}^2$.

Definition : Strict Stationarity

A stochastic process $(\epsilon_i)_{i\in\mathbb{Z}}$ is said to be strictly stationary if the joint distributions of $(\epsilon_{t_1}, \ldots, \epsilon_{t_k})$ and $(\epsilon_{t_1+h}, \ldots, \epsilon_{t_k+h})$ are the same for all positive integers k and for all $t_1, \ldots, t_k, h \in \mathbb{Z}$.

We define a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$: $\mathcal{F}_i = \sigma(\epsilon_k, k \leq i)$.

Spectral density

Autocovariance function of the error process:

$$\gamma(k) = \mathrm{Cov}(\epsilon_m, \epsilon_{m+k}) = \mathbb{E}(\epsilon_m \epsilon_{m+k}),$$

and the covariance matrix: $\Gamma_n = [\gamma(j-l)]_{1 \leq j,l \leq n}$.
Let f be the associated spectral density, that is the positive function on $[-\pi, \pi]$ such that:

$$\gamma(k) = \int_{-\pi}^{\pi} e^{ik\lambda} f(\lambda) d\lambda$$

Sommaire

Hannan's condition on the error process

Given the design X, Hannan (1973) has proved a CLT in the stationary case for the usual LSE $\hat{\beta}$ under very mild conditions.

- $\forall j \in \mathbb{Z}$ and $\forall Z \in \mathbb{L}^2(\Omega)$: $P_j(Z) = \mathbb{E}(Z|\mathcal{F}_j) - \mathbb{E}(Z|\mathcal{F}_{j-1})$.
- **Hannan's condition** on the error process:

$$\sum_{i \geq 0} \|P_0(\epsilon_i)\|_{\mathbb{L}^2} < +\infty.$$

This implies: $\sum_k |\gamma(k)| < +\infty$.

Hannan's condition is satisfied for most of short-range dependent processes.

Examples which verify Hannan's condition

- Linear Processes (Dedecker, Merlevède, Vólny (2007))
- Functions of linear processes (DMV)
- Conditions à la Gordin (DMV)
- Framework of Wu (Wu (2005))
- Weakly dependent sequences (Dedecker-Prieur (2004), Caron-Dede (2017))

Hannan's conditions on the design

- Let $X_{.,j}$ be the column j of the matrix X, $j \in \{1, \dots, p\}$:

$$d_j(n) = \|X_{.,j}\|_2 = \sqrt{\sum_{i=1}^{n} x_{i,j}^2},$$

and let $D(n)$ be the diagonal matrix with diagonal term $d_j(n)$.

- **Conditions on the design**:

$$\forall j \in \{1, \dots, p\}, \qquad \lim_{n \to \infty} d_j(n) = \infty \qquad a.s.,$$

$$\forall j \in \{1, \dots, p\}, \qquad \lim_{n \to \infty} \frac{\sup_{1 \leq i \leq n} |x_{i,j}|}{d_j(n)} = 0 \qquad a.s.,$$

and the following limits exist:

$$\forall j, l \in \{1, \dots, p\}, k \in \{0, \dots, n-1\}, \quad \rho_{j,l}(k) = \lim_{n \to \infty} \sum_{m=1}^{n-k} \frac{x_{m,j} x_{m+k,l}}{d_j(n) d_l(n)} \; a.s.$$

We define the $p \times p$ matrix $R(k)$:

$$R(k) = [\rho_{j,l}(k)] = \int_{-\pi}^{\pi} e^{ik\lambda} F_X(d\lambda) \qquad a.s.,$$

where F_X is the spectral measure associated with the matrix $R(k)$.

Moreover, we suppose:

$$R(0) > 0 \qquad a.s.$$

Then let F and G be the matrices:

$$F = \frac{1}{2\pi} \int_{-\pi}^{\pi} F_X(d\lambda) \qquad a.s.,$$

$$G = \frac{1}{2\pi} \int_{-\pi}^{\pi} F_X(d\lambda) \otimes f(\lambda) \qquad a.s.$$

Theorem (Hannan (1973))

Under the previous conditions, for all bounded continuous function f:

$$\mathbb{E}\left(f\left(D(n)(\hat{\beta}-\beta)\right)\Big|X\right) \xrightarrow[n\to\infty]{a.s.} \mathbb{E}\left(f(Z)\Big|X\right),$$

where the distribution of Z given X is: $\mathcal{N}(0, F^{-1}GF^{-1})$.
Furthermore we have the convergence of second order moment:

$$\mathbb{E}\left(D(n)(\hat{\beta}-\beta)(\hat{\beta}-\beta)^t D(n)^t\Big|X\right) \xrightarrow[n\to\infty]{a.s.} F^{-1}GF^{-1}.$$

Remark

Let us notice that, by the dominated convergence theorem, we have for any bounded continuous function f:

$$\mathbb{E}\left(f\left(D(n)(\hat{\beta}-\beta)\right)\right) \xrightarrow[n\to\infty]{} \mathbb{E}\left(f(Z)\right).$$

Sommaire

To obtain confidence regions or test procedures, one needs to estimate the limiting covariance matrix $F^{-1}GF^{-1}$. By Hannan, we have:

$$\mathbb{E}\left(D(n)(\hat{\beta}-\beta)(\hat{\beta}-\beta)^t D(n)^t \middle| X\right) \xrightarrow[n\to\infty]{a.s.} F^{-1}GF^{-1},$$

and:

$$\mathbb{E}\left(D(n)(\hat{\beta}-\beta)(\hat{\beta}-\beta)^t D(n)^t \middle| X\right) = D(n)(X^tX)^{-1}X^t\Gamma_n X(X^tX)^{-1}D(n),$$

with $\Gamma_n = [\gamma(j-l)]_{1\leq j,l\leq n}$ (covariance matrix of the error process).
Then we need an estimator of Γ_n.

Let us first consider a preliminary random matrix:

$$\widehat{\Gamma}_{n,h_n} = \left[K\left(\frac{j-l}{h_n}\right) \hat{\gamma}_{j-l} \right]_{1 \leq j,l \leq n}$$

with:

$$\hat{\gamma}_k = \frac{1}{n} \sum_{j=1}^{n-|k|} \epsilon_j \epsilon_{j+|k|}, \qquad 0 \leq |k| \leq (n-1).$$

The function K is a kernel such that:

- K is nonnegative, symmetric, and $K(0) = 1$
- K has compact support
- the fourier transform of K is integrable.

The sequence of positive reals h_n is such that $h_n \xrightarrow[n\to\infty]{} \infty$ and $\frac{h_n}{n} \xrightarrow[n\to\infty]{} 0$.

In our context, $(\epsilon_i)_{i\in\{1,\dots,n\}}$ is not observed. Only the residuals are available:

$$\hat{\epsilon}_i = Y_i - (x_i)^t\hat{\beta} = Y_i - \sum_{j=1}^{p} x_{i,j}\hat{\beta}_j,$$

because only the data Y and the design X are observed. Consequently, we consider the following estimator of Γ_n:

$$\widehat{\Gamma}^*_{n,h_n} = \left[K\left(\frac{j-l}{h_n}\right)\hat{\gamma}^*_{j-l}\right]_{1\leq j,l\leq n}$$

where:

$$\hat{\gamma}^*_k = \frac{1}{n}\sum_{j=1}^{n-|k|}\hat{\epsilon}_j\hat{\epsilon}_{j+|k|}, \qquad 0\leq|k|\leq(n-1).$$

To estimate the asymptotic covariance matrix $F^{-1}GF^{-1}$, we use the estimator:

$$C_n = D(n)(X^tX)^{-1}X^t\widehat{\Gamma}^*_{n,h_n}X(X^tX)^{-1}D(n).$$

Let us denote by C the matrix $F^{-1}GF^{-1}$ and the coefficients of the matrices C_n and C are respectively denoted by $c_{n,(j,l)}$ and $c_{j,l}$, for all j, l in $1, ..., p$.

Consistence

Theorem (C. (2019))

Let h_n be a sequence of positive reals such that $h_n \to \infty$ as n tends to infinity, and:

$$h_n \mathbb{E}\left(|\epsilon_0|^2 \left(1 \wedge \frac{h_n}{n} |\epsilon_0|^2 \right) \right) \xrightarrow[n\to\infty]{} 0.$$

Then, under the assumptions of Hannan's Theorem, the estimator C_n is consistent, that is for all j, l in $1, ..., p$:

$$\mathbb{E}\left(\left| c_{n,(j,l)} - c_{j,l} \right| \middle| X \right) \xrightarrow[n\to\infty]{} 0$$

Remark

If $\epsilon_0 \in \mathbb{L}^2$, *then there exists* $h_n \to \infty$ *such that* $h_n \mathbb{E}\left(|\epsilon_0|^2 \left(1 \wedge \frac{h_n}{n} |\epsilon_0|^2\right)\right) \xrightarrow[n\to\infty]{} 0$ *holds.*
In particular, if ϵ_0 *has a fourth order moment, then the condition is verified if* $\frac{h_n}{\sqrt{n}} \to 0$.

Corollary

Under the same conditions, the estimator C_n *converges in probability to* C *as* n *tends to infinity.*

Sketch of the proof

Let $V(X)$ be the matrix $\mathbb{E}\left(D(n)(\hat{\beta}-\beta)(\hat{\beta}-\beta)^t D(n)^t \middle| X\right)$, and let $v_{j,l}$ be its coefficients. By the triangle inequality, $\forall j,l \in \{1,\dots,p\}$:

$$\left|c_{n,(j,l)} - c_{j,l}\right| \leq \left|v_{j,l} - c_{j,l}\right| + \left|c_{n,(j,l)} - v_{j,l}\right|.$$

Thanks to Hannan's Theorem:

$$\lim_{n\to\infty} \mathbb{E}\left(\left|v_{j,l} - c_{j,l}\right| \middle| X\right) = 0, \quad a.s.$$

Then it remains to prove that:

$$\lim_{n\to\infty} \mathbb{E}\left(\left|c_{n,(j,l)} - v_{j,l}\right| \middle| X\right) = 0, \quad a.s.$$

We have:

$$V(X) = D(n)(X^tX)^{-1}X^t\Gamma_n X(X^tX)^{-1}D(n)$$

$$C_n = D(n)(X^tX)^{-1}X^t\widehat{\Gamma}^*_{n,h_n} X(X^tX)^{-1}D(n)$$

Thanks to the convergence of $D_n(X^tX)^{-1}D_n$ to $R(0)^{-1}$, it is sufficient to consider the matrices:

$$V' = D_n^{-1} X^t \Gamma_n X D_n^{-1}, \qquad C'_n = D_n^{-1} X^t \widehat{\Gamma}^*_{n,h_n} X D_n^{-1}.$$

We know that $\Gamma_n = \sum_{k=-n+1}^{n-1} \gamma(k) J_n^{(k)}$. Thus we have:

$$D(n)^{-1} X^t \Gamma_n X D(n)^{-1} = \sum_{k=-n+1}^{n-1} \gamma(k) B_{k,n}$$

$$D(n)^{-1} X^t \widehat{\Gamma}^*_{n,h_n} X D(n)^{-1} = \sum_{k=-n+1}^{n-1} K\left(\frac{k}{h_n}\right) \hat{\gamma}^*_k B_{k,n}$$

with $B_{k,n} = D(n)^{-1} X^t J_n^{(k)} X D(n)^{-1}$.

$$\left| c'_{n,(j,l)} - v'_{j,l} \right| = \left| \sum_{k=-n+1}^{n-1} \left(K\left(\frac{k}{h_n}\right) \hat{\gamma}^*_k - \gamma(k) \right) b_{j,l}^{k,n} \right|$$

where $b_{j,l}^{k,n}$ is the coefficient (j,l) of the $B_{k,n}$ matrix.

Then:

$$\sum_{k=-n+1}^{n-1} \left(K\left(\frac{k}{h_n}\right) \hat{\gamma}_k^* - \gamma(k) \right) B_{k,n} = \int_{-\pi}^{\pi} \left(f_n^*(\lambda) - f(\lambda)\right) g_n(\lambda)(d\lambda)$$

with:

$$g_n(\lambda) = \frac{1}{2\pi} \sum_{k=-(n-1)}^{n-1} e^{ikx} B_{k,n}.$$

in such a way that the matrices $B_{k,n}$ are the Fourier coefficients of the function $g_n(\lambda)$:

$$B_{k,n} = \int_{-\pi}^{\pi} e^{ik\lambda} g_n(\lambda) d\lambda.$$

Thus it remains to prove that, for all j, l in $\{1, \ldots, p\}$:

$$\lim_{n\to\infty} \mathbb{E}\left(\left| \int_{-\pi}^{\pi} \left(f_n^*(\lambda) - f(\lambda)\right) g_n(\lambda)_{j,l} d\lambda \right| \Big| X \right) = 0, \quad a.s.$$

We have:

$$\mathbb{E}\left(\left|\int_{-\pi}^{\pi}\left(f_n^*(\lambda)-f(\lambda)\right)\left[g_n(\lambda)\right]_{j,l}d\lambda\right|\Big|X\right)$$

$$\leq \sup_{\lambda\in[-\pi,\pi]}\mathbb{E}\left(\left|f_n^*(\lambda)-f(\lambda)\right|\Big|X\right)\int_{-\pi}^{\pi}\left|\left[g_n(\lambda)\right]_{j,l}\right|d\lambda$$

because $[g_n(\lambda)]_{j,l}$ is measurable with respect to the σ-algebra generated by the design X. And:

$$\sup_{\lambda\in[-\pi,\pi]}\mathbb{E}\left(\left|f_n^*(\lambda)-f(\lambda)\right|\Big|X\right)\int_{-\pi}^{\pi}\left|\left[g_n(\lambda)\right]_{j,l}\right|d\lambda$$

$$\leq \sup_{\lambda\in[-\pi,\pi]}\mathbb{E}\left(\left|f_n^*(\lambda)-f(\lambda)\right|\Big|X\right).$$

Proof: Spectral density estimate

Then consider the following estimator:

$$f_n^*(\lambda) = \frac{1}{2\pi} \sum_{|k|\leq n-1} K\left(\frac{|k|}{h_n}\right) \hat{\gamma}_k^* e^{ik\lambda}, \qquad \lambda \in [-\pi, \pi],$$

where:

$$\hat{\gamma}_k^* = \frac{1}{n} \sum_{j=1}^{n-|k|} \hat{\epsilon}_j \hat{\epsilon}_{j+|k|}, \qquad 0 \leq |k| \leq (n-1).$$

$(f(\lambda) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma(k) e^{ik\lambda})$.

Theorem (C.-Dede (2018))

Let h_n be a sequence of positive integers such that $h_n \to \infty$ as n tends to infinity, and: $h_n \mathbb{E}\left(|\epsilon_0|^2 \left(1 \wedge \frac{h_n}{n}|\epsilon_0|^2\right)\right) \xrightarrow[n\to\infty]{} 0$. Then, under the assumptions of Hannan's Theorem:

$$\sup_{\lambda\in[-\pi,\pi]} \|f_n^*(\lambda) - f(\lambda)\|_{\mathbb{L}^1} \xrightarrow[n\to\infty]{} 0.$$

This theorem is true for a fixed design X. But a quick look to the proof of this theorem suffices to see that:

$$\lim_{n\to\infty} \sup_{\lambda\in[-\pi,\pi]} \mathbb{E}\left(|f_n^*(\lambda) - f(\lambda)| \,\middle|\, X\right) = 0, \quad a.s.$$

Corollary (Hannan's theorem + Consistence theorem)

Corollary

Under the assumptions of Hannan's Theorem and the previous theorem (Consistence of C_n), we get:

$$C_n^{-\frac{1}{2}} \left(D(n)(\hat{\beta} - \beta) \right) \xrightarrow[n \to \infty]{\mathcal{L}} \mathcal{N}(0, I_p),$$

where I_p is the $p \times p$ identity matrix.

Consequently, we can obtain confidence regions and tests for β in this dependent context.

Sommaire

"Fisher" test: Dependent case

$H_0 : \beta_{j_1} = \ldots = \beta_{j_{p_0}} = 0$, against $H_1 : \exists j_z \in \{j_1, \ldots, j_{p_0}\}$ such that $\beta_{j_z} \neq 0$. If the error process is strictly stationary, we have:

$$C_{n_{p_0}}^{-1/2} \begin{pmatrix} d_{j_1}(n)(\hat{\beta}_{j_1} - \beta_{j_1}) \\ \vdots \\ d_{j_{p_0}}(n)(\hat{\beta}_{j_{p_0}} - \beta_{j_{p_0}}) \end{pmatrix} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0_{p_0\times 1}, I_{p_0}),$$

Then under H_0-hypothesis:

$$\begin{pmatrix} Z_{1,n} \\ \vdots \\ Z_{p_0,n} \end{pmatrix} = C_{n_{p_0}}^{-1/2} \begin{pmatrix} d_{j_1}(n)\hat{\beta}_{j_1} \\ \vdots \\ d_{j_{p_0}}(n)\hat{\beta}_{j_{p_0}} \end{pmatrix} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0_{p_0\times 1}, I_{p_0}),$$

and we define the following test statistic:

$$\Xi = Z_{1,n}^2 + \cdots + Z_{p_0,n}^2.$$

Under the H_0-hypothesis, $\Xi \xrightarrow[n\to\infty]{\mathcal{L}} \chi_{p_0}^2$.

One-parameter test

If we have $H_0 : \beta_j = 0$ against $H_1 : \beta_j \neq 0$, for j in $\{1, \ldots, p\}$ ("Student" test), under the H_0-hypothesis:

$$d_j(n)\hat{\beta}_j \xrightarrow[n\to\infty]{} \mathcal{N}(0, c_{j,j})$$

Then the test statistic is:

$$T_{j,n} = \frac{d_j(n)\hat{\beta}_j}{\sqrt{c_{n,(j,j)}}}$$

Under the H_0-hypothesis, $T_{j,n} \xrightarrow[n\to\infty]{} \mathcal{N}(0, 1)$

Simulations

Goal: Tests with estimated level at 5%.

$$C_n = D(n)(X^tX)^{-1}X^t\widehat{\Gamma}^*_{n,h_n}X(X^tX)^{-1}D(n),$$

with:

$$\widehat{\Gamma}^*_{n,h_n} = \left[K\left(\frac{j-l}{h_n}\right)\hat{\gamma}^*_{j-l}\right]_{1\leq j,l\leq n}$$

For the kernel K, we shall use:

$$\left\{\begin{array}{lcll} K(x) & = & 1 & if\ |x|\leq 0.8 \\ K(x) & = & 5-5|x| & if\ 0.8\leq|x|\leq 1 \\ K(x) & = & 0 & if\ |x|>1. \end{array}\right.$$

This kernel verifies the conditions to apply the consistence theorem. It is close to the rectangular kernel (whose Fourier transform is not integrable). Hence, the parameter h_n can be understood as the number of covariance terms that are necessary to obtain a good approximation of Γ_n. To choose its values, we shall use the graph of the empirical autocovariance of the residuals.

Example

We first simulate $(Z_1, \ldots, Z_n)$ according to the $AR(1)$ equation $Z_{k+1} = \frac{1}{2}(Z_k + \eta_{k+1})$, where:

- Z_1 is uniformly distributed over $[0, 1]$
- $(\eta_i)_{i \geq 2}$ is a sequence of i.i.d. random variables with distribution $\mathcal{B}(1/2)$, independent of Z_1.

Let us define:

$$\epsilon_i = F^{-1}_{\mathcal{N}(0,\sigma^2)}(Z_i).$$

By construction, ϵ_i is $\mathcal{N}(0, \sigma^2)$-distributed (but the sequence $(\epsilon_i)_{i \geq 1}$ is not a Gaussian process). For this process, one can show that Hannan's condition is satisfied because the process is $\tilde{\phi}$-dependent (in the sense of Dedecker-Prieur). But it is not α-mixing in the sense of Rosenblatt.

For the simulations, σ^2 is chosen equal to 25.

First model

First model simulated:

$$Y_i = \beta_0 + \beta_1(i^2 + X_i) + \epsilon_i, \quad \forall i \in \{1, \dots, n\}$$

with $(X_i)_{i\geq 1}$ a gaussian $AR(1)$ process (the variance is equal to 9), independent of the process $(\epsilon_i)_{i\geq 1}$.
We test H_0: $\beta_1 = 0$, against H_1: $\beta_1 \neq 0$.

- $\beta_0 = 3$.
- Under H_0, the same Fischer test is carried out 2000 times. Then we look at the frequency of rejection of the test (under H_0), that is to say the estimated level of the test (we want an estimated level close to 5%).

- Case $\beta_1 = 0$ and $h_n = 1$ (no correction):

n	200	400	600	800	1000
Estimated level	0.203	0.195	0.183	0.205	0.202

Here, since $h_n = 1$, we do not estimate any of the covariance terms. The result is that the estimated levels are too large. The test will reject the null hypothesis too often.

The parameter h_n may be chosen by analyzing the graph of the empirical autocovariances. For this example, the shape of the empirical autocovariance suggests to keep only 4 terms. This leads to choose $h_n = 5$.

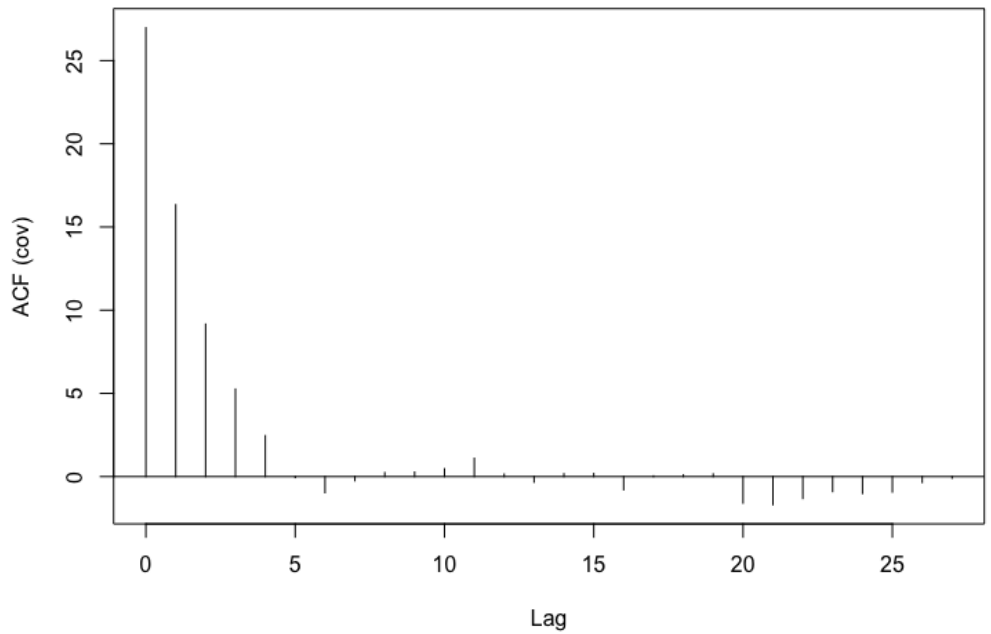


Figure : Empirical autocovariances of the residuals.

- Case $\beta_1 = 0$, $h_n = 5$:

n	200	400	600	800	1000
Estimated level	0.0845	0.065	0.0595	0.054	0.053

As suggested by the graph of the empirical autocovariances, the choice $h_n = 5$ gives a better estimated level than $h_n = 1$. If $n = 2000$ the estimated level is around 0.05.

• Case $\beta_1 = 0.00001$, $h_n = 5$:
In this example, H_0 is not satisfied. We perform the same tests as above ($N = 2000$) to estimate the power of the test.

n	200	400	600	800	1000
Estimated power	0.1025	0.301	0.887	1	1

As one can see, the estimated power is always greater than 0.05, as expected. Still as expected, the estimated power increases with the size of the samples. As soon as $n = 800$, the test always rejects the H_0-hypothesis.

Second model

$$Y_i = \beta_0 + \beta_1(\log(i) + \sin(i) + X_i) + \beta_2 i + \epsilon_i, \quad \forall i \in \{1, ..., n\}$$

We test H_0: $\beta_1 = \beta_2 = 0$ against H_1: $\beta_1 \neq 0$ or $\beta_2 \neq 0$. The coefficient β_0 is equal to 3, and we use the same simulation scheme as above.

- Case $\beta_1 = \beta_2 = 0$ and $h_n = 1$ (no correction):

n	200	400	600	800	1000
Estimated level	0.348	0.334	0.324	0.3295	0.3285

As for the first simulation, if $h_n = 1$ the test will reject the null hypothesis too often.

As suggested by the graph of the estimated autocovariances, it suggests to keep only 5 terms of covariances. This leads to choose $h_n = 6.25$.

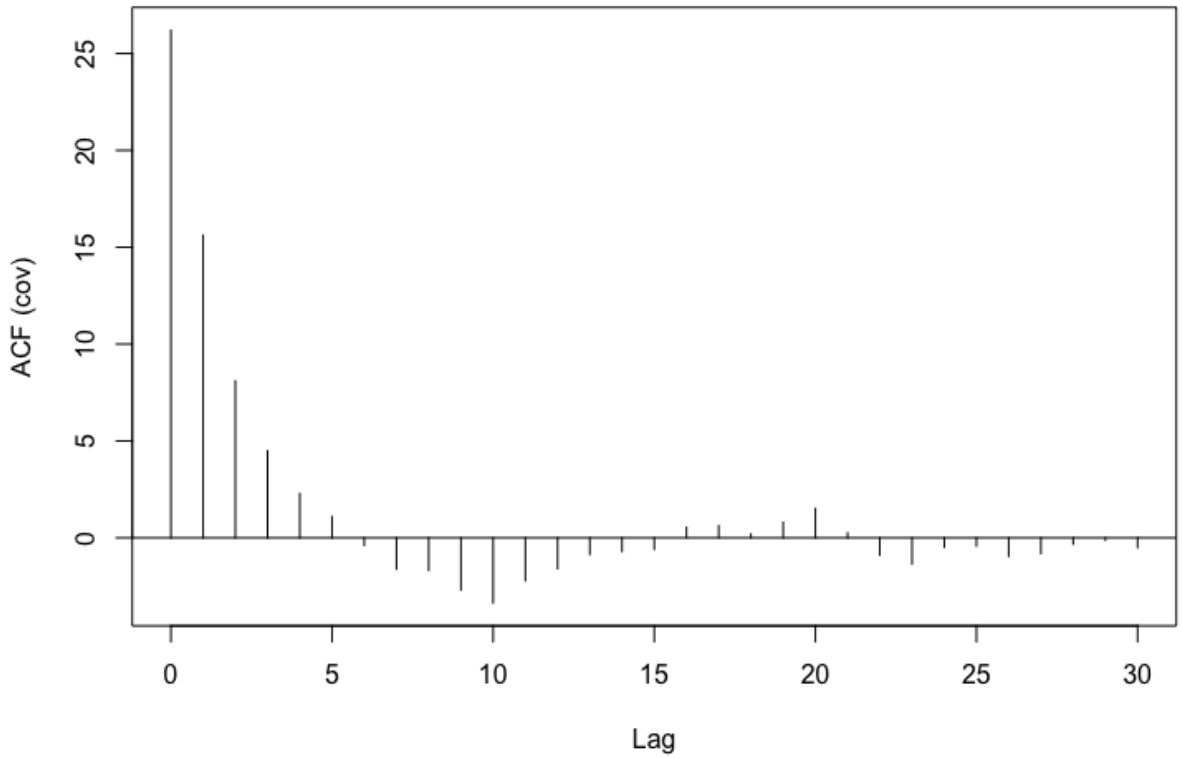


Figure : Empirical autocovariances of the residuals.

- Case $\beta_1 = \beta_2 = 0$, $h_n = 6.25$:

n	200	400	600	800	1000
Estimated level	0.09	0.078	0.066	0.0625	0.0595

Here, we see that the choice $h_n = 6.25$ works well. For $n = 1000$, the estimated level is around 0.06. If $n = 2000$ and $h_n = 6.25$, the estimated level is around 0.05.

- Case $\beta_1 = 0.2$, $\beta_2 = 0$, $h_n = 6.25$:

Now, we study the estimated power of the test.

n	200	400	600	800	1000
Estimated power	0.33	0.5	0.6515	0.776	0.884

As expected, the estimated power increases with the size of the samples, and it is around 0.9 when $n = 1000$.

Choice of h_n

Goal: Tests with estimated level at 5%.

$$C_n = D(n)(X^tX)^{-1}X^t\widehat{\Gamma}^*_{n,h_n}X(X^tX)^{-1}D(n),$$

with:

$$\widehat{\Gamma}^*_{n,h_n} = \left[K\left(\frac{j-l}{h_n}\right)\hat{\gamma}^*_{j-l}\right]_{1\leq j,l\leq n}$$

How to choose the bandwidth h_n?
Developing a package R for the applications, with automatic methods:

- Fit an AR process on the residuals. Replace the covariance matrix by the theoretical covariance matrix of the AR process.
- Bootstrap
- Other methods (based on the spectral density)

Perspectives

- Model Selection in a dependent framework: begin with linear regression model, with stationary gaussian errors.
- To consider the case where p (number of variables) is greater than n (number of observations)

Thank you !