# The regression models with dependent errors

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# Introduction
## Time Series: CO2

A typical example to show the correlations between the observations:

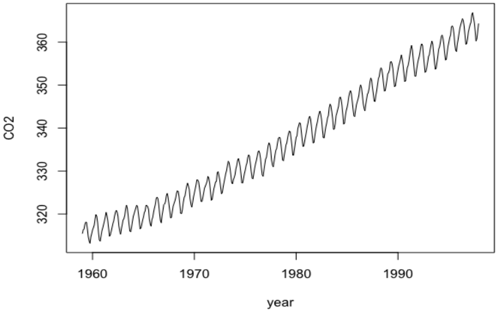



Figure: CO2 rate as a function of time.

# From Time Series to Linear Regression Model

$$Y_t = \underbrace{trend + seasonality}_{deterministic} + \underbrace{errors}_{random}.$$

Then:

$$Y = X\beta + \epsilon,$$

with:

$$X = \begin{pmatrix} 1 & 1^2 & 1^3 & \cos(\frac{2\pi}{3}) & \sin(\frac{2\pi}{3}) & \dots & \cos(\frac{2\pi}{12}) & \sin(\frac{2\pi}{12}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ t & t^2 & t^3 & \cos(\frac{2\pi t}{3}) & \sin(\frac{2\pi t}{3}) & \dots & \cos(\frac{2\pi t}{12}) & \sin(\frac{2\pi t}{12}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ n & n^2 & n^3 & \cos(\frac{2\pi n}{3}) & \sin(\frac{2\pi n}{3}) & \dots & \cos(\frac{2\pi n}{12}) & \sin(\frac{2\pi n}{12}) \end{pmatrix}$$

# ACF of the residuals

$\hat{\beta} = (X^t X)^{-1} X^t Y$: Least Squares Estimators, $\hat{\epsilon} = Y - \hat{Y}$: residuals.

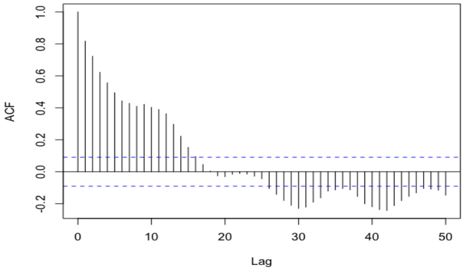



Figure: Autocorrelation of the residuals.

This is important for the applications to consider the dependency of the error process.

# Goals

1. Investigate the linear regression model in the case where the errors are dependent
2. Modification of the usual results (confidence intervals, tests, . . . ). Focus on the "Fisher's test" and its calibration
3. Study the non-parametric regression model in the case where the errors are Gaussian and dependent, via a model selection approach.

# Summary

1. Some definitions
2. Presentation of Hannan's Theorem (1973) [11]: convergence of the LSE in the stationary case under very mild conditions
3. Estimation of the asymptotic covariance matrix
4. Application: modification and calibration of the "Fisher's tests"
5. Gaussian non-parametric regression.

# Some definitions

## Linear model

Let us consider the regression linear model:

$$Y = X\beta + \epsilon,$$

where:

- $X$ is a random or deterministic design, matrix of size $[n \times p]$
- $Y$ is a $n$ random vector of observations
- $\beta$ is the $p$ vector of unknown parameters
- $\epsilon$ are the errors and $\epsilon \in \mathbb{R}^n$. In the following, the error process is independent of the design $X$.

# Least Squares Estimators

Let us recall the definition of the Least Squares Estimators (LSE):

$$\hat{\beta} = \mathrm{argmin}_{\beta \in \mathbb{R}^p} \|Y - X\beta\|_2^2 = (X^t X)^{-1} X^t Y,$$

($\|.\|_2$= euclidean norm).

We have:

- $\hat{Y} = X\hat{\beta}$: Orthogonal Projection of $Y$ on $\mathcal{M}_X = Vect\{X_{.,1}, \ldots, X_{.,p}\}$
- Residual vector: $\hat{\epsilon} = Y - \hat{Y} = Y - X\hat{\beta} \in \mathcal{M}_X^{\perp}$
- $\hat{\sigma}^2 = \frac{\|\hat{\epsilon}\|_2^2}{n-p}$.

# Strict Stationarity

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.
The error process $(\epsilon_i)_{i\in\mathbb{Z}}$ is defined on $(\Omega, \mathcal{F}, \mathbb{P})$, supposed strictly stationary with zero mean, and $\epsilon_0 \in \mathbb{L}^2(\Omega)$.

**Definition: Strict Stationarity**

A stochastic process $(\epsilon_i)_{i\in\mathbb{Z}}$ is said to be strictly stationary if the joint distributions of $(\epsilon_{t_1}, \ldots, \epsilon_{t_k})$ and $(\epsilon_{t_1+h}, \ldots, \epsilon_{t_k+h})$ are the same for all positive integers $k$ and for all $t_1, \ldots, t_k, h \in \mathbb{Z}$.

Let $(\mathcal{F}_i)_{i\in\mathbb{Z}}$ be a non-decreasing filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ defined as follows: $\mathcal{F}_i = \sigma(\epsilon_k, k \leq i)$.

# Spectral density

Let us define the autocovariance function of the error process:

$$\gamma(k) = \mathrm{Cov}(\epsilon_m, \epsilon_{m+k}) = \mathbb{E}(\epsilon_m \epsilon_{m+k}),$$

and we denote by $\Gamma_n$ the toeplitz covariance matrix:
$\Gamma_n = [\gamma(j-l)]_{1 \leq j,l \leq n}$.

Let $f$ be the associated spectral density, that is the positive function on $[-\pi, \pi]$ such that:

$$\gamma(k) = \int_{-\pi}^{\pi} e^{ik\lambda} f(\lambda) d\lambda.$$

# Hannan's theorem

## Condition on the error process

In the following, we work conditionally at design $X$. Given $X$, Hannan (1973) [11] has proved a Central Limit Theorem in the stationary case for the usual LSE $\hat{\beta}$ under very mild conditions.

- Let $(P_j)_{j\in\mathbb{Z}}$ be a family of projection operators: $\forall j \in \mathbb{Z}$ and $\forall Z \in \mathbb{L}^2(\Omega)$: $P_j(Z) = \mathbb{E}(Z|\mathcal{F}_j) - \mathbb{E}(Z|\mathcal{F}_{j-1})$.
- **Hannan's condition** on the error process:

$$\sum_{i\geq 0} \|P_0(\epsilon_i)\|_{\mathbb{L}^2} < +\infty.$$

This implies the short memory: $\sum_k |\gamma(k)| < +\infty$.
**Hannan's condition is satisfied for most of short-range dependent processes.** (Linear Processes, Functions of linear processes (Dedecker, Merlevède, Volný (2007) [7]), Weakly dependent sequences (Dedecker and Prieur (2005) [8], Caron and Dede (2018) [4]), ...).

# Hannan's conditions on the design

Let $X_{.,j}$ be the column $j$ of the matrix $X$, $j \in \{1, \dots, p\}$, and $d_j(n)$ the euclidean norm of $X_{.,j}$: $d_j(n) = \|X_{.,j}\|_2 = \sqrt{\sum_{i=1}^n x_{i,j}^2}$.
Let $D(n)$ be the diagonal normalization matrix with diagonal term $d_j(n)$.

**Conditions on the design**:

- $\forall j \in \{1, \dots, p\}, \qquad \lim_{n\to\infty} d_j(n) = \infty \qquad a.s.$
- $\forall j \in \{1, \dots, p\}, \qquad \lim_{n\to\infty} \frac{\sup_{1\le i\le n}|x_{i,j}|}{d_j(n)} = 0 \qquad a.s.,$

  and the following limits exist, $\forall j, l \in \{1, \dots, p\}, k \in \{0, \dots, n-1\}$:

- $\rho_{j,l}(k) = \lim_{n\to\infty} \sum_{m=1}^{n-k} \frac{x_{m,j}x_{m+k,l}}{d_j(n)d_l(n)} \quad a.s.$

# Hannan's theorem

## Theorem (Hannan (1973) [11])

*Under the previous conditions, for all bounded continuous function $f$:*

$$\mathbb{E}\left(f\left(D(n)(\hat{\beta}-\beta)\right)\middle|X\right)\xrightarrow[n\to\infty]{a.s.}\mathbb{E}\left(f(Z)\middle|X\right),$$

*where the distribution of $Z$ given $X$ is: $\mathcal{N}(0,C)$. Furthermore we have the convergence of second order moment:*

$$\mathbb{E}\left(D(n)(\hat{\beta}-\beta)(\hat{\beta}-\beta)^t D(n)^t\middle|X\right)\xrightarrow[n\to\infty]{a.s.}C.$$

## Remark

*Let us notice that, by the dominated convergence theorem, we have for any bounded continuous function f:*
$\mathbb{E}\left(f\left(D(n)(\hat{\beta}-\beta)\right)\right)\xrightarrow[n\to\infty]{}\mathbb{E}\left(f(Z)\right).$

# Estimation of the covariance matrix

To obtain confidence regions or test procedures, one needs to estimate the limiting covariance matrix $C$. By Hannan, we have:

$$\mathbb{E}\left(D(n)(\hat{\beta}-\beta)(\hat{\beta}-\beta)^t D(n)^t \middle| X\right) \xrightarrow[n\to\infty]{a.s.} C,$$

and:

$$\mathbb{E}\left(D(n)(\hat{\beta}-\beta)(\hat{\beta}-\beta)^t D(n)^t \middle| X\right) = D(n)(X^tX)^{-1}X^t\Gamma_n X(X^tX)^{-1}D(n),$$

with $\Gamma_n = [\gamma(j-l)]_{1\leq j,l\leq n}$ (covariance matrix of the error process).

**Consequently, we only need an estimator of $\Gamma_n$.**

# Residual-based Kernel estimator

Let us consider the following estimator of $\Gamma_n$:

$$\widehat{\Gamma}^*_{n,h_n} = \left[ K\left(\frac{j-l}{h_n}\right) \hat{\gamma}^*_{j-l} \right]_{1 \leq j,l \leq n},$$

with[2] : $\hat{\gamma}^*_k = \frac{1}{n} \sum_{j=1}^{n-|k|} \hat{\epsilon}_j \hat{\epsilon}_{j+|k|}, \qquad 0 \leq |k| \leq (n-1).$

The function $K$ is a kernel such that:

- K is nonnegative, symmetric, and $K(0) = 1$
- K has compact support
- the fourier transform of $K$ is integrable.

The sequence of positive reals $h_n$ is such that $h_n \to \infty$ and $\frac{h_n}{n} \to 0$ as $n \to \infty$.

[2] In our context, $(\epsilon_i)_{i \in \{1,\ldots,n\}}$ is not observed. Only the residuals $\hat{\epsilon}_i = Y_i - (x_i)^t \hat{\beta}$ are available, because only the data $Y$ and the design $X$ are observed.

# Covariance matrix estimator

To estimate the asymptotic covariance matrix $C$, we use the estimator:

$$C_n = D(n)(X^t X)^{-1} X^t \widehat{\Gamma}^*_{n,h_n} X (X^t X)^{-1} D(n).$$

The coefficients of the matrices $C_n$ and $C$ are respectively denoted by $c_{n,(j,l)}$ and $c_{j,l}$ , for all $j, l$ in $\{1, ..., p\}$.

# Consistency

The following theorem proves, under mild conditions, the $\mathbb{L}^1$-norm consistency given $X$ of the covariance matrix estimator:

**Theorem (C. (2019) [3])**

*Let $h_n$ be a sequence of positive reals such that $h_n \to \infty$ as $n \to \infty$, and:*

$$h_n \mathbb{E}\left(|\epsilon_0|^2 \left(1 \wedge \frac{h_n}{n} |\epsilon_0|^2\right)\right) \xrightarrow[n\to\infty]{} 0.$$

*Then, under the assumptions of Hannan's Theorem, the estimator $C_n$ is consistent, that is for all $j, l$ in $\{1, ..., p\}$:*

$$\mathbb{E}\left(\left|c_{n,(j,l)} - c_{j,l}\right| \middle| X\right) \xrightarrow[n\to\infty]{} 0.$$

# $h_n$ condition

## Corollary

*Under the same conditions, the estimator $C_n$ converges in probability to $C$ as $n$ tends to infinity.*

The condition:

$$h_n \mathbb{E}\left(|\epsilon_0|^2\left(1 \wedge \frac{h_n}{n}|\epsilon_0|^2\right)\right) \xrightarrow[n\to\infty]{} 0. \tag{1}$$

is a very general condition.

## Remark

*If $\epsilon_0 \in \mathbb{L}^2$, then there exists $h_n \to \infty$ such that (1) holds.*
*In particular, if $\epsilon_0$ has a fourth order moment, then the condition is verified if $\frac{h_n}{\sqrt{n}} \to 0$.*

# Sketch of the proof

Let $V(X)$ be the matrix $\mathbb{E}\left(D(n)(\hat{\beta}-\beta)(\hat{\beta}-\beta)^t D(n)^t \middle| X\right)$, and let $v_{j,l}$ be its coefficients. By the triangle inequality, $\forall j,l \in \{1,\dots,p\}$:

$$\left|c_{n,(j,l)} - c_{j,l}\right| \leq \left|v_{j,l} - c_{j,l}\right| + \left|c_{n,(j,l)} - v_{j,l}\right|.$$

Thanks to Hannan's Theorem:

$$\lim_{n\to\infty} \mathbb{E}\left(\left|v_{j,l} - c_{j,l}\right| \middle| X\right) = 0, \quad a.s.$$

**Then it remains to prove that:**

$$\lim_{n\to\infty} \mathbb{E}\left(\left|c_{n,(j,l)} - v_{j,l}\right| \middle| X\right) = 0, \quad a.s.$$

We have:

$$V(X) = D(n)(X^tX)^{-1}X^t\Gamma_n X(X^tX)^{-1}D(n)$$

$$C_n = D(n)(X^tX)^{-1}X^t\widehat{\Gamma}^*_{n,h_n}X(X^tX)^{-1}D(n).$$

Thanks to the convergence of $D_n(X^tX)^{-1}D_n$ (Hannan's conditions), it is sufficient to consider the matrices:

$$V' = D_n^{-1}X^t\Gamma_n X D_n^{-1}, \qquad C_n' = D_n^{-1}X^t\widehat{\Gamma}^*_{n,h_n}XD_n^{-1}.$$

We know that $\Gamma_n = \sum_{k=-n+1}^{n-1}\gamma(k)J_n^{(k)}$, where $J_n^{(k)}$ is a matrix with some $1$ on the $k$-th diagonal. Thus we have:

$$D(n)^{-1}X^t\Gamma_n XD(n)^{-1} = \sum_{k=-n+1}^{n-1}\gamma(k)B_{k,n}$$

$$D(n)^{-1}X^t\widehat{\Gamma}^*_{n,h_n}XD(n)^{-1} = \sum_{k=-n+1}^{n-1}K\left(\frac{k}{h_n}\right)\hat{\gamma}^*_k B_{k,n},$$

with $B_{k,n} = D(n)^{-1}X^tJ_n^{(k)}XD(n)^{-1}$.

$$\left|c'_{n,(j,l)} - v'_{j,l}\right| = \left|\sum_{k=-n+1}^{n-1}\left(K\left(\frac{k}{h_n}\right)\hat{\gamma}_k^* - \gamma(k)\right) b_{j,l}^{k,n}\right|,$$

where $b_{j,l}^{k,n}$ is the coefficient $(j,l)$ of the $B_{k,n}$ matrix.

We recall that:

$$f(\lambda) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\gamma(k)e^{ik\lambda}, \qquad \gamma(k) = \int_{-\pi}^{\pi} e^{ik\lambda} f(\lambda)d\lambda,$$

and:

$$f_n^*(\lambda) = \frac{1}{2\pi}\sum_{k=-n+1}^{n-1} K\left(\frac{|k|}{h_n}\right)\hat{\gamma}_k^* e^{ik\lambda}, \qquad K\left(\frac{|k|}{h_n}\right)\hat{\gamma}_k^* = \int_{-\pi}^{\pi} e^{ik\lambda} f_n^*(\lambda)d\lambda.$$

Then:

$$\sum_{k=-n+1}^{n-1} \left( K\left(\frac{k}{h_n}\right) \hat{\gamma}_k^* - \gamma(k) \right) B_{k,n} = \int_{-\pi}^{\pi} \left(f_n^*(\lambda) - f(\lambda)\right) g_n(\lambda)(d\lambda),$$

with:

$$g_n(\lambda) = \frac{1}{2\pi} \sum_{k=-(n-1)}^{n-1} e^{ik\lambda} B_{k,n},$$

in such a way that the matrices $B_{k,n}$ are the Fourier coefficients of the function $g_n(\lambda)$:

$$B_{k,n} = \int_{-\pi}^{\pi} e^{ik\lambda} g_n(\lambda) d\lambda.$$

Thus it remains to prove that, for all $j, l$ in $\{1, \ldots, p\}$:

$$\lim_{n\to\infty} \mathbb{E}\left( \left| \int_{-\pi}^{\pi} \left(f_n^*(\lambda) - f(\lambda)\right) [g_n(\lambda)]_{j,l} d\lambda \right| \middle| X \right) = 0, \quad a.s.$$

We have:

$$\mathbb{E}\left(\left|\int_{-\pi}^{\pi}(f_n^*(\lambda)-f(\lambda))\,[g_n(\lambda)]_{j,l}d\lambda\right|\Big|X\right)$$
$$\leq \sup_{\lambda\in[-\pi,\pi]}\mathbb{E}\left(|f_n^*(\lambda)-f(\lambda)|\Big|X\right)\int_{-\pi}^{\pi}\left|[g_n(\lambda)]_{j,l}\right|d\lambda,$$

because $[g_n(\lambda)]_{j,l}$ is measurable with respect to the $\sigma$-algebra generated by the design X.

Then, we can prove that:

$$\int_{-\pi}^{\pi}\left|[g_n(\lambda)]_{j,l}\right|d\lambda \leq 1.$$

Consequently:

$$\sup_{\lambda\in[-\pi,\pi]}\mathbb{E}\left(|f_n^*(\lambda)-f(\lambda)|\Big|X\right)\int_{-\pi}^{\pi}\left|[g_n(\lambda)]_{j,l}\right|d\lambda$$
$$\leq \sup_{\lambda\in[-\pi,\pi]}\mathbb{E}\left(|f_n^*(\lambda)-f(\lambda)|\Big|X\right).$$

# Proof: Spectral density estimate

Let us consider the following estimator of the spectral density, for $\lambda \in [-\pi, \pi]$: $f_n^*(\lambda) = \frac{1}{2\pi} \sum_{|k| \leq n-1} K\left(\frac{|k|}{h_n}\right) \hat{\gamma}_k^* e^{ik\lambda}$, where:
$\hat{\gamma}_k^* = \frac{1}{n} \sum_{j=1}^{n-|k|} \hat{\epsilon}_j \hat{\epsilon}_{j+|k|}, \qquad 0 \leq |k| \leq (n-1).$

**Theorem (C. and Dede (2018) [4])**

*Under the same assumptions of the consistency Theorem:*

$$\sup_{\lambda \in [-\pi,\pi]} \|f_n^*(\lambda) - f(\lambda)\|_{\mathbb{L}^1} \xrightarrow[n \to \infty]{} 0.$$

This theorem has been proved for a fixed design $X$, but it remains true with a random design:

$$\lim_{n \to \infty} \sup_{\lambda \in [-\pi,\pi]} \mathbb{E}\left(|f_n^*(\lambda) - f(\lambda)| \,\middle|\, X\right) = 0, \quad a.s.$$

The proof is complete.

# Corollary (Hannan's theorem + Consistency theorem)

**Corollary**

*Under the assumptions of Hannan's Theorem and the consistency theorem (Consistency of $C_n$), we get:*

$$C_n^{-\frac{1}{2}} \left( D(n)(\hat{\beta} - \beta) \right) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0, I_p),$$

*where $I_p$ is the $p \times p$ identity matrix.*

Consequently, we can obtain confidence regions and tests for $\beta$ in this dependent context.

# Tests

- We are interested in test procedures on the linear model, particularly the "Fisher's" tests
- Thanks to the previous corollary, we can establish a new test statistic, so that the tests on the linear model always have an asymptotically good level, even when the underlying error process is dependent
- The level of a test (denoted by $\alpha$) is the probabilty to choose $H_1$ hypothesis while $H_0$ is true.

## "Fisher's" test: Dependent case

$H_0 : \beta_{j_1} = \ldots = \beta_{j_{p_0}} = 0$, against $H_1 : \exists j_z \in \{j_1, \ldots, j_{p_0}\}$ such that $\beta_{j_z} \neq 0$. If the error process is strictly stationary, we have:

$$C_{n_{p_0}}^{-1/2} \begin{pmatrix} d_{j_1}(n)(\hat{\beta}_{j_1} - \beta_{j_1}) \\ \vdots \\ d_{j_{p_0}}(n)(\hat{\beta}_{j_{p_0}} - \beta_{j_{p_0}}) \end{pmatrix} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0_{p_0\times 1}, I_{p_0}).$$

Then under $H_0$-hypothesis:

$$\begin{pmatrix} Z_{1,n} \\ \vdots \\ Z_{p_0,n} \end{pmatrix} = C_{n_{p_0}}^{-1/2} \begin{pmatrix} d_{j_1}(n)\hat{\beta}_{j_1} \\ \vdots \\ d_{j_{p_0}}(n)\hat{\beta}_{j_{p_0}} \end{pmatrix} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0_{p_0\times 1}, I_{p_0}),$$

and we define the following test statistic: $\Xi = Z_{1,n}^2 + \cdots + Z_{p_0,n}^2$. Under the $H_0$-hypothesis, $\Xi \xrightarrow[n\to\infty]{\mathcal{L}} \chi^2_{p_0}$.
In the same way, we can define an univariate test.

# Bandwidth calibration

We have defined test procedures with a level asymptotically equal to $\alpha$ ($\alpha$ to be determined, typically $5\%$).

**Question:** With a finite value of observations, how to choose the bandwidth $h_n$ in order to have well-calibrated tests and a non-asymptotic level as close as possible to the wanted level $\alpha$ ?

Two main difficulties in our context:

1. Our target is the level of a test, which differs from classical approaches where the risk of an estimator is considered
2. We are not only in a context of dependent variables, but also in the very general framework of Hannan whose theorem applies for most stationary short-memory processes.

Consequently we can not use directly the classical methods of adaptive statistics in our framework.

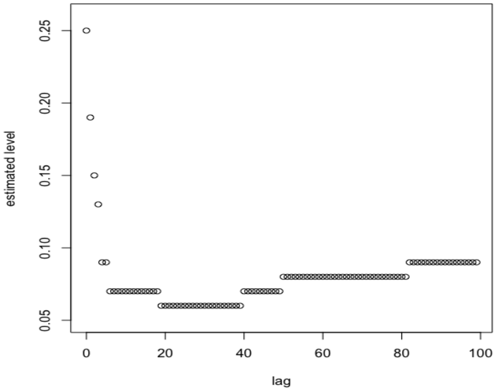



Figure: Level curve: estimated level as a function of the lag; $n = 1000$ observations, $T = 100$ simulations.

# Empirical methods

- It is of first importance to provide hypothesis tests with correct significance levels
- We need data driven methods for the applications
- We partially answered to this problem by constructing empirical methods based on the data
- We propose a "plug-in" approach which consists in replacing the estimator of $\Gamma_n$. So we introduce the following estimator:

$$\widehat{C} = \widehat{C}(\widehat{\Gamma}_n) := D(n)(X^t X)^{-1} X^t \widehat{\Gamma}_n X (X^t X)^{-1} D(n),$$

and we use $\widehat{C}$ to compute the usual statistics of the linear model.

We have defined different ways to obtain the $\widehat{\Gamma}_n$ matrix:

1. by adapting an autoregressive process on the residual process and computing the theoretical covariances of the obtained AR(p) process. The order of the AR process is chosen by an AIC criterion
2. using the kernel estimator defined in Caron [3] with a bootstrap method to choose the value of the window (Wu and Pourahmadi (2009) [15])
3. by using an alternative choice of the window for the rectangular kernel (Efromovich (1998) [9])
4. in using an adaptive estimator of the spectral density via a histogram base (Comte (2001) [6]), with the slope heuristic algorithm to choose the dimension.

All these methods have been programmed on **R** in the *slm* package available on the CRAN.

# Simulations

Let us define the three following processes:

1. AR(1) process (called "AR1"): $\epsilon_i - 0.7\epsilon_{i-1} = W_i$, where $W_i \sim \mathcal{N}(0,1)$
2. MA(12) process (called "MA12"): $\epsilon_i = W_i + 0.5W_{i-2} + 0.3W_{i-3} + 0.2W_{i-12},$, where the $(W_i)$'s are i.i.d. random variables following Student's distribution with $10$ degrees of freedom
3. A dynamical system (called "Sysdyn"): for $\gamma \in ]0,1[$, the intermittent map $\theta_\gamma : [0,1] \mapsto [0,1]$ introduced by Liverani, Saussol and Vaienti [12] is defined by

$$\theta_\gamma(x) = \begin{cases} x(1+2^\gamma x^\gamma) & \text{if} \quad x \in [0,1/2[ \\ 2x-1 & \text{if} \quad x \in [1/2,1]. \end{cases}$$

The Sysdyn process is then defined by $\varepsilon_i = \theta_\gamma^i$ (For the simulations, $\gamma = 1/4$). It is a non-mixing process (in the sense of Rosenblatt), with an arithmetic decay of the correlations ($\sim \frac{1}{k^3}$ if $\gamma = 1/4$).

- Let us define the following linear regression model, for all $i$ in $\{1, \ldots, n\}$ ($\beta_1 = 3$ and $Z_i$ is a gaussian AR(1) process):

$$Y_i = \beta_1 + \beta_2(\log(i) + \sin(i) + Z_i) + \beta_3 i + \varepsilon_i$$

- We simulate a $n$-error process according to the AR1, the MA12 or the Sysdyn processes (small samples ($n = 200$) and larger ($n = 1000, 5000$))
- We simulate realizations of the linear regression model under the null hypothesis: $H_0 : \beta_2 = \beta_3 = 0$
- We make the test like described above
- The simulations are repeated $1000$ times.

| n | Method / Process | Fisher i.i.d. | fitAR | spectral proj | efromovich | kernel |
|---|---|---|---|---|---|---|
| 200 | AR1 process | 0.465 | **0.097** | 0.14 | 0.135 | 0.149 |
| | Sysdyn process | 0.385 | **0.105** | 0.118 | 0.124 | 0.162 |
| | MA12 process | 0.228 | **0.113** | **0.113** | 0.116 | 0.15 |
| 1000 | AR1 process | 0.418 | 0.043 | **0.049** | **0.049** | 0.086 |
| | Sysdyn process | 0.393 | **0.073** | 0.077 | 0.079 | 0.074 |
| | MA12 process | 0.209 | 0.064 | 0.066 | 0.069 | **0.063** |
| 5000 | AR1 process | 0.439 | 0.044 | **0.047** | **0.047** | **0.047** |
| | Sysdyn process | 0.381 | 0.058 | 0.061 | **0.057** | 0.064 |
| | MA12 process | 0.242 | 0.044 | **0.048** | 0.043 | 0.057 |

Table: Estimated levels.

# Gaussian model selection theorem in a dependent context

- Estimation of a non-random vector $f^* \in \mathbb{R}^n$ in the model:

$$Y = f^* + \epsilon, \quad \text{where } \epsilon \sim \mathcal{N}(0_{n\times 1}, \Sigma_{n\times n})$$

- Study the regression model in the non-parametric case via a model selection approach
- Develop a model selection theory with penalization in the framework of Gaussian dependent variables
- Establish an oracle inequality for the minimal risk estimator among a collection of models
- For short range and long range dependent Gaussian processes

# Framework

- Estimation of a non-random vector $f^* \in \mathbb{R}^n$ from the observation $Y$, in the model

$$Y = f^* + \epsilon, \quad \text{where } \epsilon \sim \mathcal{N}(0_{n\times 1}, \Sigma_{n\times n})$$

- $\Sigma$ is the $n \times n$ covariance matrix with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$. The spectral radius of $\Sigma$

$$\rho(\Sigma) = \max_{1\leq i\leq n} \lambda_i = \lambda_1$$

- Let $\{S_m, m \in \mathcal{M}\}$ be a collection of finite-dimensional spaces, with $d_m = dim(S_m)$
- $\hat{f}_m = \mathrm{Proj}_{S_m}^{\perp} Y$ is the least squares estimator of $f^*$ on $S_m$. It minimizes the contrast function

$$\gamma_n(t) = \|Y - t\|_n^2, \quad \forall t \in S_m$$

($\|.\|_n$ : normalized euclidean norm in $\mathbb{R}^n$)

- $\ell^2$-risk of an estimator $\hat{f}_m$

$$R(\hat{f}_m) = \mathbb{E}\left[\left\|\hat{f}_m - f^*\right\|_n^2\right]$$

- Using Pythagoras equality, we have the bias-variance decomposition

$$\mathbb{E}\left[\left\|f^* - \mathrm{Proj}_{S_m}(Y)\right\|_n^2\right] = \left\|(\mathrm{Id} - \mathrm{Proj}_{S_m})f^*\right\|_n^2 + \mathbb{E}\left[\left\|\mathrm{Proj}_{S_m}(\varepsilon)\right\|_n^2\right]$$

  We can prove that the variance term is equal to

$$\mathbb{E}\left[\left\|\mathrm{Proj}_{S_m}(\varepsilon)\right\|_n^2\right] = \frac{1}{n}\,\mathrm{tr}(\mathrm{Proj}_{S_m}\,\Sigma)$$

  Usually, bias and variance have opposite behaviors according to the dimension.

- We want to find the dimension that balances bias and variance, and select the oracle estimator $\hat{f}_{m_0}$ such that

$$m_0 \in \mathrm{argmin}_{m\in\mathcal{M}}\{R(\hat{f}_m)\}$$

- The true risk is unknown in practice, then we introduce the empirical risk
$$\widehat{R}(\hat{f}_m) = \left\| Y - \hat{f}_m \right\|_n^2$$
- This typically leads to overfitting, then we have to penalize the larger models.
- **Aim** : select a model in the collection such that the risk of the selected estimator is as close as possible to the oracle model
$$\hat{m} \in \mathrm{argmin}_{m \in \mathcal{M}} \left\{ \left\| Y - \hat{f}_m \right\|_n^2 + \mathrm{pen}(m) \right\},$$
where $\mathrm{pen} : \mathcal{M} \to \mathbb{R}^+$ is a penalty function
- We perform a non asymptotic analysis of the risk of the selected estimator $\hat{f}_{\hat{m}}$ in the dependent Gaussian context

# A general Gaussian model selection result

Let $\pi = \{\pi_m, m \in \mathcal{M}\}$ be a distribution of probability on $\mathcal{M}$ associated with the collection of models $\{S_m, m \in \mathcal{M}\}$, such that $\sum_{m\in\mathcal{M}} \pi_m = 1$

**Theorem (C., Dedecker and Michel (2020))**

*Let $K > 1$, and let $\mathrm{pen} : \mathcal{M} \to \mathbb{R}^+$ be a penalty function such that, for any $m \in \mathcal{M}$,*

$$\mathrm{pen}(m) \geq \frac{K}{n}\left(\sqrt{\mathrm{tr}\left(\mathrm{Proj}_{S_m}\Sigma\right) + \rho(\Sigma)} + \sqrt{\rho(\Sigma)}\sqrt{2\log\left(\frac{1}{\pi_m}\right)}\right)^2 .$$

*Then there exists a constant $C > 1$ which only depends on $K$ such that the estimator $\hat{f}_{\hat{m}}$ selected satisfies*

$$\mathbb{E}\left[\left\|f^* - \hat{f}_{\hat{m}}\right\|_n^2\right] \leq C\left(\inf_{m\in\mathcal{M}}\left\{\mathbb{E}\left[\left\|f^* - \hat{f}_m\right\|_n^2\right] + \mathrm{pen}(m)\right\} + \frac{\rho(\Sigma)}{n}\right).$$

# Proof - Some key points

**Theorem (Inequality from Cirel'son, Ibragimov and Sudakov [5].)**

*Let $F : (\mathbb{R}^n, \|\cdot\|) \to \mathbb{R}$ be a 1-Lipschitz function and $\eta$ a random vector in $\mathbb{R}^n$ such that $\eta \sim \mathcal{N}_n(0, \sigma^2 Id)$ for some $\sigma > 0$. Then there exists a random variable $\xi$ following an exponential distribution of parameter 1 such that*

$$F(\eta) \leq \mathbb{E}\left[F(\eta)\right] + \sigma\sqrt{2\xi}.$$

**Lemma**

*Let $\Sigma$ be a $n \times n$ symmetric semidefinite matrix and $S$ a linear subspace of $\mathbb{R}^n$. Let $\varepsilon$ be a Gaussian random vector such that $\varepsilon \sim \mathcal{N}_n(0, \Sigma)$. Then there exists a random variable $\xi$ following an exponential distribution of parameter 1 such that*

$$\left\|\mathrm{Proj}_S(\varepsilon)\right\|_n \leq \mathbb{E}\left\|\mathrm{Proj}_S(\varepsilon)\right\|_n + \sqrt{\frac{\rho(\Sigma)}{n}}\sqrt{2\xi}.$$

## Proof of Lemma.

Let $\varepsilon \sim \mathcal{N}_n(0, \Sigma)$, then $\varepsilon$ satisfies $\varepsilon = \sqrt{\Sigma}\eta$ with $\eta \sim \mathcal{N}_n(0, Id)$. Let $S$ be a linear subspace of $\mathbb{R}^n$. We then check that the function $\eta \to \left\|\mathrm{Proj}_S(\sqrt{\Sigma}\eta)\right\|_n$ is a Lipschitz function

$$\begin{aligned}\left\|\mathrm{Proj}_S(\sqrt{\Sigma}x) - \mathrm{Proj}_S(\sqrt{\Sigma}y)\right\|_n &\leq \left\|\sqrt{\Sigma}(x-y)\right\|_n \\ &\leq \rho(\sqrt{\Sigma})\left\|x-y\right\|_n \\ &\leq \sqrt{\rho(\Sigma)}\left\|x-y\right\|_n = \sqrt{\frac{\rho(\Sigma)}{n}}\left\|x-y\right\|.\end{aligned}$$

By applying the theorem to the function $\eta \to \left\|\mathrm{Proj}_S(\sqrt{\Sigma}\eta)\right\|_n$, we find that

$$\left\|\mathrm{Proj}_S(\sqrt{\Sigma}\eta)\right\|_n \leq \mathbb{E}\left\|\mathrm{Proj}_S(\sqrt{\Sigma}\eta)\right\|_n + \sqrt{\frac{\rho(\Sigma)}{n}}\sqrt{2\xi}.$$

□

$$\mathrm{pen}(m) \geq \frac{K}{n}\left(\sqrt{\mathrm{tr}\left(\mathrm{Proj}_{S_m}\Sigma\right) + \rho(\Sigma)} + \sqrt{\rho(\Sigma)}\sqrt{2\log\left(\frac{1}{\pi_m}\right)}\right)^2$$

- The main term in the penalty shape is the trace term $\mathrm{tr}\left(\mathrm{Proj}_{S_m}\Sigma\right)$
- It plays the same role as the term $\mathrm{Var}(\varepsilon_1)d_m$ in the results of Birgé and Massart for independent Gaussian errors [1, 13]
- This penalty can only be calculated if the matrix $\Sigma$ is completely known. However, in certain cases, we can consider effective strategies to circumvent this issue

# Short range dependent case

- We have an easier penalty shape from the upper bound

$$\mathrm{tr}\left(\mathrm{Proj}_{S_m}\Sigma\right) \leq d_m \rho(\Sigma)$$

- With a minor modification of the proof of the previous theorem, the risk bound

$$\mathbb{E}\left[\left\|f^* - \hat{f}_{\hat{m}}\right\|_n^2\right] \leq C\left(\inf_{m\in\mathcal{M}}\left\{\mathbb{E}\left[\left\|f^* - \hat{f}_m\right\|_n^2\right] + \mathrm{pen}(m)\right\} + \frac{\rho(\Sigma)}{n}\right)$$

  is still valid when

$$\mathrm{pen}(m) \geq K\frac{\rho(\Sigma)}{n}\left(\sqrt{d_m} + \sqrt{2\log\left(\frac{1}{\pi_m}\right)}\right)^2, \text{ for any } K > 1$$

- If the sequence $(\varepsilon_i)_{i\geq 1}$ is a stationary and short memory Gaussian process, then the spectral radius is bounded and the penalty shape is completely in line with the case of i.i.d. Gaussian errors [1, 13]
- The usual variance term $\mathrm{Var}(\varepsilon_1)$ has been replaced by the spectral radius $\rho(\Sigma)$.
- If the collection of model is not too rich, then

$$\mathrm{pen}(m) \sim K' \rho(\Sigma) d_m$$

In practice, the penalty can be chosen proportional to the model dimension m and calibrated according to the slope heuristic method introduced by Birgé et Massart [2]

# Slope heuristic

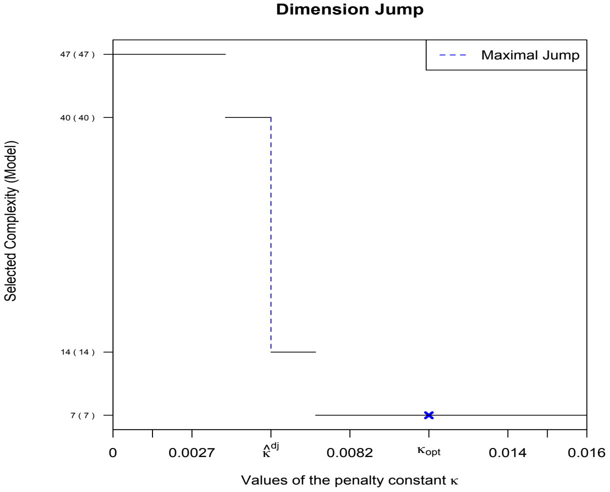
- To calibrate the penalty function, we use the slope heuristics method proposed by Birgé and Massart [2].
- The aim is to tune the constant $\kappa$ in a penalty of the form $\mathrm{pen}(m) = \kappa\,\mathrm{pen}_{\text{shape}}(m)$ (in the most standard cases, $\mathrm{pen}_{\text{shape}}$ is the dimension of the model). Let $\hat{m}(\kappa)$ be the model selected by the penalized criterion with constant $\kappa$

$$\hat{m}(\kappa) \in \mathrm{argmin}_{m \in \mathcal{M}} \left\{ \frac{1}{n} \left\| Y - \hat{f}_m \right\|_n^2 + \kappa\,\mathrm{pen}_{\text{shape}}(m) \right\}$$

The Dimension Jump algorithm consists of the following steps

1. Compute $\kappa \mapsto \hat{m}(\kappa)$,
2. Find the constant $\hat{\kappa}^{dj} > 0$ that corresponds to the highest jump of the function $\kappa \to d_{\hat{m}(\kappa)}$,
3. Select the model $\hat{m}(2\hat{\kappa}^{dj})$,

$$\hat{m} \in \mathrm{argmin}_{m \in \mathcal{M}} \left\{ \left\| Y - \hat{f}_m \right\|_n^2 + 2\hat{\kappa}^{dj}\,\mathrm{pen}_{\text{shape}}(m) \right\}$$


Dimension Jump
Maximal Jump
47 ( 47 )
40 ( 40 )
14 ( 14 )
7 ( 7 )
Selected Complexity (Model)
0
0.0027
$\hat{\kappa}^{dj}$
0.0082
$\kappa_{opt}$
0.014
0.016
Values of the penalty constant $\kappa$

Figure

# Long range dependent case

- It is tempting to keep this penalty shape as a general penalty shape for Gaussian linear model selection with dependent errors. However, this is too rough in some cases (for instance for long range dependent processes)

- When the error process is a long range dependent Gaussian process, the spectral radius of the covariance matrix is not bounded

$\Rightarrow$ the previous selection model procedure is not working !

- An other penalty shape must be defined

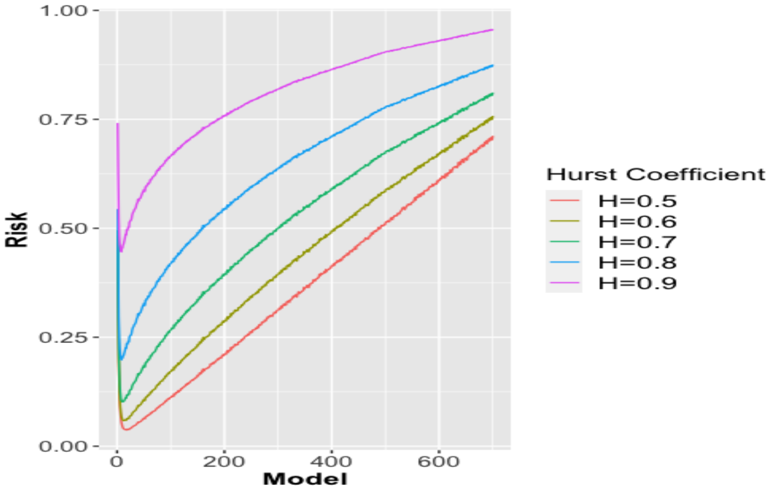



Figure: Comparison of risk shapes for the fractional Gaussian process with Hurst coefficient between 0.5 and 0.9, and for $n = 2000$.

- We shall only consider here the linear spaces $S_m$ of $\mathbb{R}^n$ generated by the family of piecewise polynomials of degree at most $r$ ($r \in \mathbb{N}$) on the regular partition of size $m$ of the interval $[0, 1]$.
- $d_m = dim(S_m) = (r + 1)m$
- The case $r = 0$ corresponds to the regular regressogram of size $m$.

- The error process $(\varepsilon_i)_{i\geq 1}$ is assumed stationary. Instead of assume that $\rho(\Sigma)$ is bounded, we assume that

$$|\gamma_\varepsilon(k)| \leq \kappa k^{-\gamma}, \quad \text{for some } \kappa > 0 \text{ and } \gamma \in (0, 1),$$

where $\gamma_\varepsilon(k) = \mathrm{Cov}(\varepsilon_0, \varepsilon_k)$

$$\mathrm{pen}(m) \geq \frac{K}{n}\left(\sqrt{\mathrm{tr}\left(\mathrm{Proj}_{S_m}\Sigma\right)+\rho(\Sigma)}+\sqrt{\rho(\Sigma)}\sqrt{2\log\left(\frac{1}{\pi_m}\right)}\right)^2$$

## Lemma

*Let $S_m$ be the linear space of $\mathbb{R}^n$ induced by the family of piecewise polynomials of degree at most $r$ on the regular partition of size $m$ of the interval $[0,1]$. Then*

$$\mathrm{tr}\left(\mathrm{Proj}_{S_m}\Sigma\right) \leq C m^{\gamma} n^{1-\gamma},$$

*where $C$ depends on $\kappa, \gamma$ and $r$.*
*Moreover, using the classical Gerschgorin theorem [10], we can proved that*

$$\rho(\Sigma_n) \leq B n^{1-\gamma},$$

*where $B$ depends on $\kappa$ and $\gamma$.*

- Using the results of this lemma, one can choose a penalty of the form

$$\mathrm{pen}(m) = K\frac{m^\gamma}{n^\gamma},$$

for some positive constant $K$ depending on $\kappa, \gamma$ and $r$

- For the applications, we would like to use the slope heuristic method. But it is necessary to estimate the parameter $\gamma$
- We propose an estimation of $\gamma$ based on the Hurst coefficient, which is estimated thanks to the Whittle estimator
- Then we use the slope heuristic method with $\mathrm{pen}_{\mathsf{shape}}(m) = m^{\hat{\gamma}}$

# Simulation with short memory ARMA

- Let $\epsilon$ be the following ARMA(2,1) gaussian process ($W_i \sim \mathcal{N}(0, 0.5)$): $\epsilon_i - 0.4\epsilon_{i-1} - 0.2\epsilon_{i-2} = W_i + 0.3W_{i-1}$
- For all $t$ in $[0, 1]$, $f^* = 3 - 0.1t + 0.5t^2 - t^3 + \sin(8t)$
- We generate a sample of size $n = 1000$, defined for all $i$ in $\{1, \ldots, n\}$:

$$Y_i = f^* \left( \frac{i}{n} \right) + \epsilon_i$$

- The goal is to adapt a regressogram and choose the best regular partition to approach the $f^*$ function.

For a dimension $m$, from $1$ to $50$, we split the interval $[0, 1]$ into $m$ intervals and the estimator $\hat{f}_m$ is a piecewise constant function, equal to the average of $Y_i$ on each interval.

This simulation is repeated $100$ times and we obtain the following mean risk curve $\left(\left\|\hat{f}_m - f^*\right\|_2^2\right)$

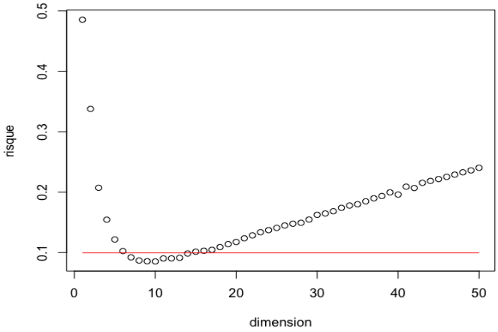



Figure: Mean risk curve for $100$ simulations, and total mean risk of the method with slope heuristic (red line).

- Evaluation of the performance of the dimension jump algorithm
- We compute the risk $\left\|\hat{f}_m - Y\right\|_2^2$
- Then we use the slope heuristic method to choose the dimension (again the simulation is repeated $100$ times).

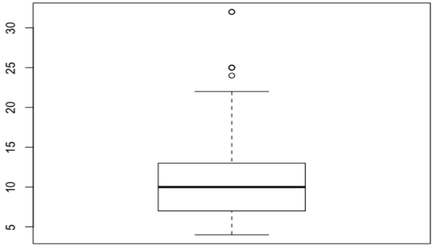



Figure: Boxplot with the dimensions selected by the dimension jump algorithm.

This represents the function $f^*$ and its estimator with a regressogram of dimension $10$ (dimension with the minimum average theoretical risk).

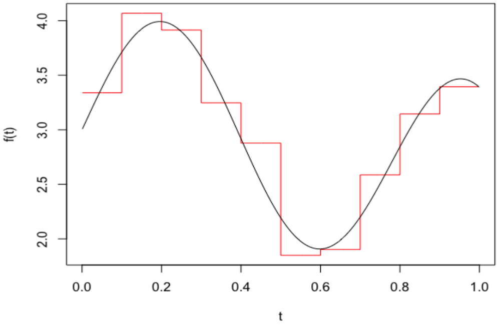



Figure: Function $f^*$ (black) and the regressogram with dimension $10$ (red).

# Applications to Nile data - long memory

The Nile data consist of readings of annual minimum levels at the Roda gorge near Cairo, commencing in the year 622; often only the first 663 observations are employed because missing observations occur after the year 1284 [14]. These data show cyclical variations, which come from a phenomenon of long memory.

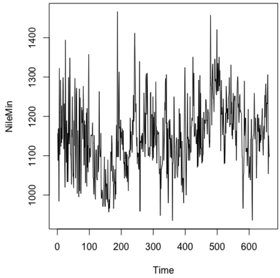
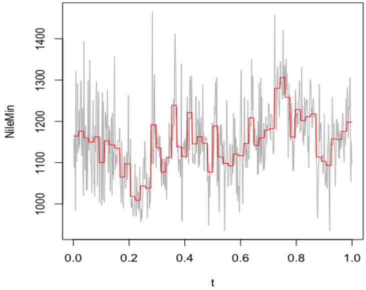
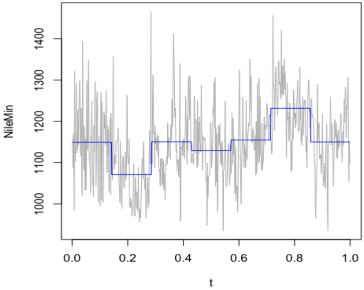



Figure: Nile River data.

We apply our methods on these data to estimate the trend, since we have a way to select automatically a partition from the data



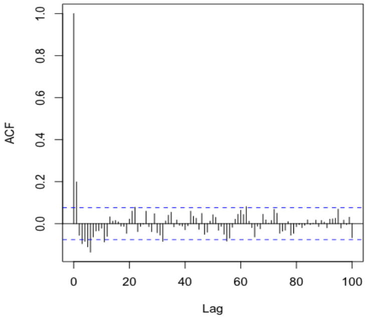

(a) The usual penalty proportional to $m$, using the "classical jump dimension" to calibrate the constant



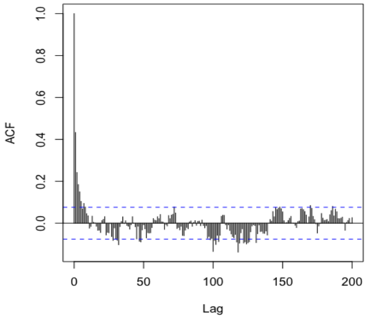

(b) A penalty proportional to $m^{\hat{\gamma}}$, where $\hat{\gamma}$ is estimated from the Hurst estimator.

Figure: Nile River data and resulting estimators.

# ACF of the residuals




(a) Penalty proportional to $m$




(b) Penalty proportional to $m^{\hat{\gamma}}$

Figure: ACF of the residuals for the two methods

# Perspectives

- Model selection
  - Non-parametric regression: generalize the previous results to the non-Gaussian case
  - Dependent Lasso
- Statistical Learning
  - Dependent variables in Statistical Learning
  - Double descent
- Spatial Statistics

**Thank you !**

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